Scaling dimensions and conformal anomaly in anisotropic lattice spin models

1987 J. Phys. A: Math. Gen. 20 L451

(http://iopscience.iop.org/0305-4470/20/7/006)

J. Phys. A: Math. Gen. **20** (1987) L451-L456. Printed in the UK

LETTER TO THE EDITOR

Scaling dimensions and conformal anomaly in anisotropic lattice spin models

Doochul Kim† and Paul A Pearce‡

† Department of Physics, Seoul National University, Seoul, 151, Korea
‡ Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

Received 24 February 1987

Abstract. The effect of anisotropic interactions on the eigenvalue spectrum of the row-to-row transfer matrix of critical lattice spin models is investigated. It is verified that the predictions of conformal theory apply to anisotropic systems if one allows for spatial rescaling by incorporating an anisotropy factor $\zeta=(a_y/a_x)\sin\theta$ where a_y and a_x are lattice spacings and θ is an angle describing the anisotropy. For exactly solvable models these anisotropy angles can be calculated analytically using corner transfer matrices. This is done for the eight-vertex model, hard hexagons and interacting hard squares and it is found that $\theta=\pi u/\lambda$, $10|u|/3$ and $5u$ respectively where u is the spectral parameter and λ is the crossing parameter. For each of these models, the amplitude of the finite-size corrections to the free energy at criticality is found to be of the form $\pi\zeta c/6N^2$ where ζ is the anisotropy factor and the central charge or conformal anomaly is given by $c=1$, $4/5$ and $7/10$ respectively. This is an analytic result for the eight-vertex model. For the hard hexagon and square models the largest eigenvalues are found accurately by numerically solving their inversion identities for various anisotropies and strip widths up to $N=48$. Finally, we argue that the anisotropy angle for magnetic hard squares and the q-state Potts models is also given by $\theta=\pi u/\lambda$, so this result is quite general.

For conformally invariant two-dimensional lattice spin models at criticality, the amplitude of the finite-size corrections to the free energy is linearly related to the conformal anomaly or central charge c that characterises different universality classes (Blöte *et al* 1986). Similarly, the scaling dimensions of scaling operators are simply related to the asymptotics of the transfer matrix eigenvalues for finite width strips (Cardy 1986, 1987). This gives a very powerful method for determining critical exponents and universality classes. Unfortunately, to be conformally invariant, a spin system must first of all be spatially isotropic. However, if anisotropic interactions are present and there is a single correlation length exponent, one can restore the isotropy and hence conformal invariance by rescaling the lattice spacings in appropriate directions (Barber *et al* 1984). In general this deforms a square unit cell into a parallelogram as shown in figure 1. The anisotropy can therefore be described by the two lattice spacings a_x, a_y and an angle θ or equivalently an anisotropy factor $\zeta=(a_y/a_x)\sin\theta$. The purpose of this letter is to report explicit calculation of θ for several solvable lattice models and to investigate the effect of anisotropy on universal finite-size corrections.

Consider a lattice made up of N columns and M rows of unit parallelograms as shown in figure 1(c). In the complex notation of Cardy (1986), the sides l, l' are given by

$$l=a_xN \qquad l'=a_yM(\sin\theta+\mathrm{i}\cos\theta). \tag{1}$$

0305-4470/87/070451+06\$02.50

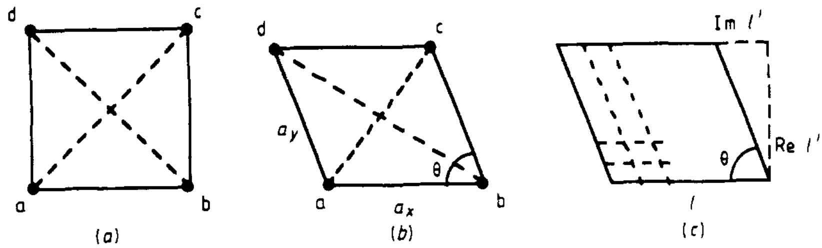

Figure 1. Rescaling of lattice spacings distorts a unit cell (a) into a parallelogram (b). Spins on each corner are denoted by a, b, c, d and full and broken lines symbolise interactions. A lattice of N columns and M rows form a parallelogram (c) of dimension $l \times l'$ where l' is complex.

Let $\Lambda_n = \exp(-E_n)$, $n = 0, 1, \ldots$ be the eigenvalues of the row-to-row transfer matrix of a strip of width N with periodic boundary conditions. Then the prediction of the theory of conformal invariance (Cardy 1986), modified to incorporate anisotropy, is that for large N

$$\begin{aligned} \mathrm{Re}\, E_n &= \frac{2\pi}{N} \frac{a_y}{a_x} \sin\theta (x_n - c/12) + Nf \\ \mathrm{Im}\, E_n &= \frac{2\pi}{N} \frac{a_y}{a_x} \cos\theta s_n + \frac{2\pi j_n}{p} \end{aligned} \tag{2}$$

where θ is the anisotropy angle, f is the bulk free energy, c is the conformal anomaly and x_n, s_n are the scaling dimension and spin respectively of the scaling operator associated with the nth eigenstate. For the largest eigenvalue Λ_0 we set $x_0 = s_0 = 0$. The phase factors $2\pi j_n/p$ ($j_n = 0, 1, \ldots, p-1$) occur when translational symmetry is broken in an adjoining ordered phase and the ordered structure repeats itself only after p translations along the y direction.

In this letter we will consider models where the anisotropic interactions preserve reflection symmetry with respect to the diagonals ac and bd of figure 1(b). This is the case when the anisotropy is due to asymmetric interactions along the diagonals. For a square lattice with nearest-neighbour interactions only, the same situation is achieved by rotating the lattice through 45°, or equivalently, by considering the diagonal-to-diagonal transfer matrix. In such cases a_x and a_y must be equal and the anisotropy is completely characterised by the single anisotropy factor $\zeta = \sin\theta$. The effective or anisotropy angle θ has been obtained for nearest-neighbour Ising models by Barber *et al* (1984) using corner transfer matrices. Here we use these methods to obtain θ for the eight-vertex and hard hexagon (square) models. The predictions of conformal invariance (2), as applied to critical anisotropic systems, will then be tested by obtaining the finite-size corrections directly.

In general, for exactly solvable models, the one-point functions can be calculated analytically using corner transfer matrices. Moreover, it is a simple consequence of conformal invariance (Cardy 1987) that, for a wedge of angle θ wrapped onto a cone with its composite edges identified, the apex exponents are given by $x_{\mathrm{apex}} = (2\pi/\theta)x$ where x is the bulk exponent. Since, for some order parameter, both x and x_{apex} can be calculated from the eigenvalues of the corner transfer matrices so can θ. We have done this for the eight-vertex and hard hexagon (square) models using the known eigenvalues of the corner transfer matrices (Baxter 1981, 1982, Baxter and Pearce 1983).

The critical eight-vertex model can be mapped (Baxter 1982) onto the critical six-vertex model with face weights parametrised as

$$\omega_1 = \sin(\lambda - u) \qquad \omega_2 = \sin u \qquad \omega_3 = \sin\lambda \qquad \omega_4 = 0 \tag{3}$$

where $0 < u < \lambda < \pi$. Similarly, the face weights of critical hard hexagons (squares) are given by

$$\begin{aligned} &\omega_1 = \sin(2\lambda + u)/\sin 2\lambda && \omega_2 = \sin u/(\sin\lambda \sin 2\lambda)^{1/2} \\ &\omega_3 = \sin(\lambda - u)/\sin\lambda && \omega_4 = \sin(2\lambda - u)/\sin 2\lambda \\ &\omega_5 = \sin(\lambda + u)/\sin\lambda \end{aligned} \tag{4}$$

where $-\lambda < u < 2\lambda$ and $\lambda = \pi/5$. The parameters u and λ are called the spectral and crossing parameters respectively. In terms of these parameters our results are

$$\theta = \pi u/\lambda \tag{5}$$

for the eight-vertex model and

$$\theta = \begin{cases} -10u/3 & -\pi/5 < u < 0 \\ 5u & 0 < u < \pi/5 \\ (10/3)(\pi/2 - u) & \pi/5 < u < 2\pi/5 \end{cases} \tag{6}$$

for the hard hexagon (square) models. Equation (5) gives the usual relation between θ and the parameters u and λ. In the latter more unusual case (6), θ is a continuous function of u with corners at $u = 0$ and $\pi/5$ where the eigenvalues cross. For pure hard hexagons ($u = -\pi/5$) we have $\theta = 2\pi/3$, consistent with the geometry of an isotropic triangular lattice (Privman and Fisher 1984). The geometrical significance of the anisotropy angles, however, is best illustrated by considering the star-triangle equations (Baxter 1982) of figure 2. Defining supplementary angles $\theta' = \pi - \theta$, the ubiquitous and somewhat mysterious constraint on the spectral parameters $u_1 + u_2 + u_3 = \lambda$ becomes the meaningful and immediately transparent statement $\theta'_1 + \theta'_2 + \theta'_3 = 2\pi$. This holds even for the anisotropy angles of the generalised hard hexagon models (6).

We will now obtain the finite-size corrections to Λ_0 by more direct means to test the validity of (2). First we consider the eight-vertex model. Recently, de Vega and Woynarovich (1985) have proposed a method for calculating finite-size corrections in non-critical systems solvable by the Bethe ansatz. Hamer (1985), and independently

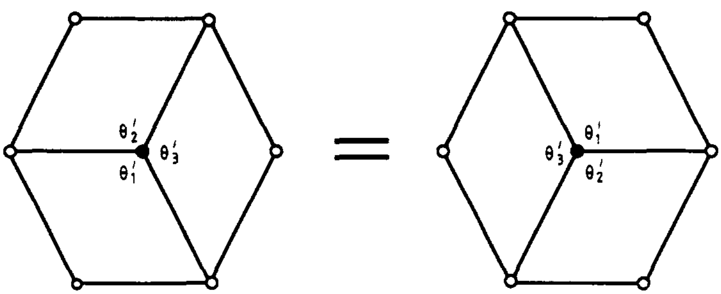

Figure 2. The star-triangle equations satisfied by solvable lattice models. The Boltzmann weights of the two graphs are equal when the centre spin is summed out. The usual constraint $u_1 + u_2 + u_3 = \lambda$ on the spectral parameters of the faces is equivalent to $\theta'_1 + \theta'_2 + \theta'_3 = 2\pi$ which states that the sum of the anisotropy angles at the centre equals 2π.

Avdeev and Dörfel (1986), extended the method to the critical case. By a slight modification of their working, the free energy

$$f_N = -(1/N)\ln \Lambda_0 = E_0/N \tag{7}$$

of the critical eight-vertex model for a strip of even width N with periodic boundary conditions is given by

$$f_N = -\frac{1}{N}\sum_{j=1}^{N/2} \ln\left(\frac{\exp(2u\mathrm{i}) - \exp(2\alpha_j - \lambda \mathrm{i})}{\exp(2\alpha_j) - \exp(2u\mathrm{i} - \lambda\mathrm{i})}\right) \tag{8}$$

where α_j are the solutions of the Bethe ansatz equation (9) of Hamer (1985). We use α and λ for his λ and γ, respectively. Following de Vega and Woynarovich (1985) now leads to an expression for the free energy correction as

$$f_N - f = -\int_{-\infty}^{\infty} \mathrm{d}\alpha \int_{-\infty}^{\infty} \mathrm{d}t \exp(-\mathrm{i}t\alpha)\frac{\sinh ut}{2t\cosh(\lambda t/2)}\left(\frac{1}{N}\sum_{j=1}^{N/2}\delta(\alpha - \alpha_j) - \sigma_N(\alpha)\right) \tag{9}$$

where $\sigma_N(\alpha)$ is defined by their equation (2.9). Using a change of integration variables as in Hamer (1985), the above expression can be integrated to give

$$f_N - f = -(\pi/6N^2)\sin(\pi u/\lambda). \tag{10}$$

Comparing with (2), we see that indeed $\theta = \pi u/\lambda$ and $c = 1$ as is characteristic of models with continuously varying exponents.

The finite-size corrections to the free energy for the hard hexagon (square) models have not yet been calculated analytically. Along the critical lines of these models, however, the eigenvalues of the row-to-row transfer matrix for a strip of width N are polynomials of degree N in the variable $\exp(2u\mathrm{i})$. Furthermore, under periodic boundary conditions, the eigenvalues satisfy the inversion identity (3.3) of Baxter and Pearce (1982). For moderately large N, this functional equation can be solved numerically for the N zeros of the polynomials from which the transfer matrix eigenvalues Λ_n can be obtained as a function of u. Table 1 shows sequences of estimators

$$\begin{aligned} c^{\mathrm{eff}}(N) &= -(6N/\pi)(E_0 - Nf) \\ x_1^{\mathrm{eff}}(N) &= (N/2\pi)\,\mathrm{Re}(E_1 - E_0) \end{aligned} \tag{11}$$

obtained in this way for the largest and next-largest eigenvalues respectively of interacting hard squares for the two typical values $u = \pi/10$ and $u = \pi/20$. Since interacting hard squares belong to the tricritical Ising model class (Huse 1982) the sequences are expected, as $N \to \infty$, to approach the limits $(7/10)\sin\theta$ and $(3/40)\sin\theta$ respectively with $\theta = 5u$. These series extend considerably those obtained by Bartelt *et al* (1986) for $N \leqslant 16$ and the isotropic case. To the right of each column in table 1, we also show the accelerated sequence obtained by one iteration of the alternating-ε algorithm (Barber *et al* 1984). The agreement of the data with the theoretical prediction (2) is excellent for the whole range of $u (0 < u < \pi/5)$ and confirms, to very high accuracy (typically 5-7 significant digits for c and 3-5 for x_1), the identifications $\theta = 5u$, $c = 7/10$ and $x_1 = 3/40$. Similarly, for hard hexagons $(-\pi/5 < u < 0,\ \pi/5 < u < 2\pi/5)$, which lies in the three-state Potts universality class, we have obtained the largest eigenvalue series for $N \leqslant 30$ and found agreement with (2) to a 4-5 digit accuracy using $c = 4/5$ (Cardy 1987) and the anisotropy angle θ given by (6).

Table 1. Sequences of estimators $c^{\text{eff}}(N)$ and $x_1^{\text{eff}}(N)$ for largest and next-largest eigenvalues respectively of interacting hard squares for the two typical values $u = \pi/10$ and $u = \pi/20$.

	$u = \pi/10$		$u = \pi/20$	
N	$c^{\text{eff}}(N)$	accelerated	$c^{\text{eff}}(N)$	accelerated
4	0.661 215 955		0.492 604 844	
8	0.688 830 406		0.495 220 940	
12	0.694 895 501	0.700 029 968	0.495 214 782	0.495 218 216
16	0.697 108 589	0.700 010 183	0.495 145 908	0.495 007 602
20	0.698 147 590	0.700 004 347	0.495 098 520	0.495 005 663
24	0.698 714 951	0.700 002 151	0.495 067 654	0.495 056 598
28	0.699 057 606	0.700 001 181	0.495 046 913	0.494 962 409
32	0.699 279 997	0.700 000 702	0.495 032 418	0.495 971 491
36	0.699 432 342	0.700 000 443	0.495 021 918	0.494 973 384
40	0.699 541 181	0.700 000 292	0.495 014 076	0.494 974 047
44	0.699 621 595		0.495 008 065	
48	0.699 682 664		0.495 003 356	
Exact		0.700 000 000		0.494 974 747
	$u = \pi/10$		$u = \pi/20$	
N	$x_1^{\text{eff}}(N)$	accelerated	$x_1^{\text{eff}}(N)$	accelerated
4	0.073 148 629		0.054 212 227	
8	0.074 939 851		0.053 779 515	
12	0.075 195 311	0.075 243 328	0.053 540 910	0.053 253 512
16	0.075 239 208	0.075 237 970	0.053 414 116	0.052 960 738
20	0.075 236 611	0.075 238 211	0.053 337 126	0.053 018 307
24	0.075 222 461	0.075 285 887	0.053 285 735	0.053 027 336
28	0.075 206 188	0.074 935 925	0.053 249 088	0.053 030 232
32	0.075 190 591	0.075 061 137	0.053 221 669	0.053 031 458
36	0.075 176 444	0.075 067 933	0.053 200 396	0.053 032 061
40	0.075 163 854	0.075 066 799	0.053 183 418	0.053 032 407
44	0.075 152 712		0.053 169 556	
48	0.075 142 849		0.053 158 027	
Exact		0.075 000 000		0.053 033 009

Another solvable model for which we have numerically determined the anisotropy angle is magnetic hard squares (Pearce 1985). We have recently shown (Pearce and Kim 1987) that the multicritical T-manifold of this model exhibits continuously varying exponents with a conformal anomaly $c = 1$. Solving the inversion identity for this model allows us to determine with high accuracy the various eigenvalue levels. Analysis of the data for strip widths up to 32 with $\theta = \pi u/\lambda$ gives excellent agreement with the theoretical prediction (2) over the complete range $0 < u < \lambda < 2\pi/3$ where the spectral and crossing parameters are defined as in Pearce (1985) and Pearce and Kim (1987).

Finally, we consider the critical q-state Potts model on the square lattice. This model is equivalent to the six-vertex model with special boundary conditions (Baxter *et al* 1976). These special boundary conditions lead to q-dependent c values in the free energy corrections (Blöte *et al* 1986). For the diagonal-to-diagonal transfer matrix, we conjecture that the anisotropy angle is again given by $\theta = \pi u/\lambda$ with the usual parametrisation $\sqrt{q} = 2\cos\lambda$ and $(\exp K_1 - 1)/\sqrt{q} = \sqrt{q}/(\exp K_2 - 1) = \sin u/\sin(\lambda - u)$ where K_1 and K_2 are the critical couplings. By a simple geometric argument (see figure 1(b)) this implies that the anisotropy factor of the *row-to-row* transfer matrix is

$$\zeta=\frac{\sin\theta}{1+\cos\theta}=\tan\frac{\theta}{2}=\tan\frac{\pi u}{2\lambda}. \tag{12}$$

In the limit $K_1\to 0$ ($\theta\to 0$), the logarithmic derivative of the row-to-row transfer matrix with respect to K_1 gives the quantum Potts Hamiltonian (von Gehlen *et al* 1986) and, working to leading order, the anisotropy factor becomes

$$\zeta=\frac{\pi}{4}\frac{\tan\lambda}{\lambda}K_1 \tag{13}$$

in agreement with the XXZ Hamiltonian representation of the quantum Potts model (Alcaraz *et al* 1987, Barber 1987).

We thank M N Barber for useful discussions. This work was done while one of us (DK) was a visitor to the University of Melbourne and the support of the Mathematics Department is gratefully acknowledged.

References

Alcaraz F C, Barber M N and Batchelor M T 1987 to be published
Avdeev L V and Dörfel B D 1986 *J. Phys. A: Math. Gen.* **19** L13-7
Barber M N 1987 private communication
Barber M N, Peschel I and Pearce P A 1984 *J. Stat. Phys.* **37** 497-527
Bartelt N C, Einstein T L and Roelofs L D 1986 *Phys. Rev.* B **34** 1616-23
Baxter R J 1981 *J. Stat. Phys.* **26** 427-52
—— 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
Baxter R J, Kelland S B and Wu F Y 1976 *J. Phys. A: Math. Gen.* **9** 397-406
Baxter R J and Pearce P A 1982 *J. Phys. A: Math. Gen.* **15** 897-910
—— 1983 *J. Phys. A: Math. Gen.* **16** 2239-55
Blöte H W J, Cardy J L and Nightingale M P 1986 *Phys. Rev. Lett.* **56** 742-5
Cardy 1986 *Nucl. Phys.* B **270** [FS16] 186-204
—— 1987 *Phase Transitions and Critical Phenomena* vol 11, ed C Domb and J L Lebowitz (New York: Academic) to be published
de Vega H J and Woynarovich F 1985 *Nucl. Phys.* B **251** [FS13] 439-56
Hamer C J 1985 *J. Phys. A: Math. Gen.* **18** L1133-7
Huse D A 1982 *Phys. Rev. Lett.* **49** 1121-4
Pearce P A 1985 *J. Phys. A: Math. Gen.* **18** 3217-26
Pearce P A and Kim D 1987 to be published
Privman V and Fisher M E 1984 *Phys. Rev.* B **30** 322-7
von Gehlen G, Rittenberg V and Ruegg H 1986 *J. Phys. A: Math. Gen.* **19** 107-19